

CONSTRUCTIONS OF GENERALIZED COMPLEX STRUCTURES IN DIMENSION FOUR

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In this note, four-manifold theory is employed to study the existence of (twisted) generalized complex structures. It is shown that there exist (twisted) generalized complex structures that have more than one type change loci. In an example-driven fashion, (twisted) generalized complex structures are constructed on a myriad of four-manifolds, both simply and non-simply connected, which are neither complex nor symplectic.

1. INTRODUCTION

Twisted generalized complex structures are a generalization of complex and symplectic structures introduced by Hitchin [20], and developed by Gualtieri in [17, 16]. The existence of an almost-complex structure is the only known obstruction for the existence of a twisted generalized complex structure on a manifold so far [16]. Given that surfaces are Kähler manifolds, the question of existence of such a structure becomes non-trivial first in dimension four. In [9, 10], Cavalcanti and Gualtieri showed that generalized complex 4-manifolds form a larger set than symplectic and/or complex manifolds. They proved that a necessary and sufficient condition for the manifolds $m\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ to admit a generalized complex structure, is that they admit an almost-complex one.

The endeavor taken in this note is to study the existence of twisted generalized complex structures using 4-manifold theory. A number of non-complex and non-symplectic twisted generalized complex manifolds are produced building on recent constructions of small symplectic 4-manifolds [6, 12, 7, 3] by using the techniques of [28, 24, 14, 4, 26, 9, 10] on Seiberg-Witten theory, symplectic and generalized complex geometry. Among the results of the paper, there are the following.

- A (twisted) generalized complex structure can have more than one type change loci.
- The connected sums

$$\begin{aligned} &m(S^2 \times S^2), r(S^2 \times S^2) \# S^3 \times S^1, \\ &m\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}, r\mathbb{CP}^2 \# s\overline{\mathbb{CP}^2} \# S^3 \times S^1, \\ &L(p, 1) \times S^1 \# k\mathbb{CP}^2 \end{aligned}$$

admit a (twisted) generalized complex structure if and only if they have an almost complex structure. In particular, the generalized complex structures

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on $m\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ built in this paper are different from the ones constructed in [10].

- Every finitely presented fundamental group is realized by a non-symplectic twisted generalized complex 4-manifold.
- Constructions of twisted generalized complex 4-manifolds that do not admit a symplectic nor a complex structure, and that have specific types of fundamental groups. For example, abelian groups, free groups of arbitrary rank, and surface groups.

The organization of the paper is the following. Section 2 contains a short introduction to twisted generalized complex structures. The main results used to equip four-manifolds with such a geometric structure, and a fundamental result on the study of the Seiberg-Witten invariants are stated in Section 3. Generalized complex structures for spin manifolds are constructed in Section 4. In Section 4.1, generalized complex structures on $S^2 \times S^2$ with different numbers of type change loci are constructed. The question of existence of a generalized complex structure on the connected sums $(2g - 3)(S^2 \times S^2)$ is settled in Section 4.2, and a preview of existence results of twisted generalized complex structures on non-simply connected manifolds is given in Section 4.3. In Section 5, a large class of symplectic 4-manifolds are put together in order to produce generalized complex structures that are neither complex nor symplectic. The sixth section is devoted to the study of the existence of these structures within the non-simply connected realm. In Section 6.1, it is proven that all finitely presented groups are twisted generalized complex, while being neither symplectic nor complex. The last part of the paper contains non-symplectic, non-complex examples of twisted generalized complex manifolds with abelian, surface and free fundamental groups (Sections 6.2 and 6.3 respectively). The paper ends with questions for further research in Section 7.

2. TWISTED GENERALIZED COMPLEX STRUCTURES

Following the work of Gualtieri in [16], in this section we recall the basic definitions and examples of generalized complex structures.

The *Courant bracket* of sections of the bundle $TM \oplus T^*M$ given by the direct sum of the tangent and cotangent bundles of a smooth manifold M is

$$[X + \xi, Y + \eta]_H := [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2}d(\eta(X) - \xi(Y)) + i_Y i_X H,$$

where H is a closed 3-form on M .

The bundle $TM \oplus T^*M$ can be equipped with a natural symmetric pairing of signature (n, n)

$$\langle X + \xi, Y + \eta \rangle := \frac{1}{2}(\eta(X) + \xi(Y))$$

as well.

Definition 1. A *twisted generalized complex structure* (M, H, \mathcal{J}) is a complex structure \mathcal{J} on the bundle $TM \oplus T^*M$ that satisfies the following two conditions.

- (1) It preserves the symmetric pairing.
- (2) Its $+i$ -eigenbundle, $L \subset T_{\mathbb{C}}M \oplus T_{\mathbb{C}}^*M$, is closed under the Courant bracket.

The $+i$ -eigenbundle L is a maximal isotropic subspace of $T_{\mathbb{C}}M \oplus T_{\mathbb{C}}^*M$ that satisfies $L \cap \bar{L} = \{0\}$. The bundle L can be used to fully describe the complex structure

\mathcal{J} . Moreover, a maximal isotropic subspace L may be uniquely described by a line bundle $K \subset \wedge^\bullet T_{\mathbb{C}}^* M$. This complex line bundle K annihilated by the $+i$ -eigenvalue of \mathcal{J} is the *canonical bundle* of \mathcal{J} .

Indeed, the twisted generalized complex structure can be completely described in terms of differential forms. In order for a complex differential form $\rho \in \wedge^\bullet T_{\mathbb{C}}^* M$ to be a local generator of the canonical line bundle K of a twisted generalized complex structure, and thus determine \mathcal{J} uniquely, it is required that ρ satisfies the following three properties at every point $p \in M$.

- Algebraic property: the form can be written as

$$\rho = e^{B+i\omega} \wedge \Omega,$$

where B, ω are real 2-forms, and Ω is a decomposable complex form.

- Non-degeneracy: the non-vanishing condition

$$(\rho, \bar{\rho}) = \Omega \wedge \bar{\Omega} \wedge (2i\omega)^{n-k} \neq 0$$

holds. Here $2n = \dim(M)$, and $k = \deg(\Omega)$.

- Integrability: the form ρ is integrable in the sense that

$$d\rho + H \wedge \rho = (X + \xi) \cdot \rho,$$

for a section $X + \xi$ of $TM \oplus T^*M$.

The non-degeneracy requirement is equivalent to the condition $L \cap \bar{L} = \{0\}$. It implies that at each point of a twisted generalized complex manifold, the real subspace $\ker \Omega \wedge \bar{\omega} \subset TM$ inherits a symplectic structure from the 2-form ω , and a transverse complex structure is defined by the annihilator of ω , as the $+i$ -eigenspace of a complex structure on $T/\ker \Omega \wedge \bar{\Omega}$.

Definition 2. *Type and parity of a twisted generalized complex structure.* Let $\rho = e^{B+i\omega} \wedge \Omega$ be a generator of the canonical bundle K of a generalized complex structure \mathcal{J} at a point $p \in M$. The *type* of \mathcal{J} at p is the degree of Ω . The *parity* of \mathcal{J} is the parity of its type.

The type of a twisted generalized complex structure need not be constant, it may jump along a codimension two submanifold. However, the parity of the type does remain the same along connected components of the manifold M .

Remark 3. *(Twisted) generalized complex structures: on the 3-form H of our constructions.* For the twisted generalized complex structures (M, H, \mathcal{J}) produced in this paper, the 3-form H is always given by a generator of $H^3(M; \mathbb{Z})$. Poincaré duality implies that if a 4-manifold is simply connected, then the 3-form satisfies $H = 0$. In this case, a generalized complex structure is obtained. In particular, the non-simply connected manifolds constructed will have a twisted generalized complex structure. For example, if the fundamental group is $\pi_1(\hat{M}) \cong \mathbb{Z}$, then the 3-form H is a generator of $H^3(M; \mathbb{Z}) \cong \mathbb{Z}$.

Two standard examples of generalized complex manifolds are the following.

Example 4. *Complex and symplectic manifolds.* Let (M^{2n}, I) be a complex manifold. Then M^{2n} is a generalized complex manifold of type n . Indeed, we can define on $TM \oplus T^*M$, the operator

$$\mathcal{J}_I := \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}.$$

In this case, $T^{0,1}M \oplus T^{*1,0}M$ is the $+i$ -eigenspace of \mathcal{J}_I , and the canonical bundle is $K = \wedge^{n,0}T^*M$.

A symplectic manifold (M, ω) is a generalized complex manifold of type 0. The operator

$$\mathcal{J}_\omega := \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

is a complex structure on the bundle $TM \oplus T^*M$, whose $+i$ -eigenspace is given by $\{X - i\omega(X) : X \in T_{\mathbb{C}}M\}$; the canonical bundle K is generated by the form $e^{i\omega}$.

Regarding the jump on the type of a twisted generalized complex structure, the following is observed. The projection

$$\wedge^\bullet T_{\mathbb{C}}^*M \rightarrow \wedge^0 T_{\mathbb{C}}^*M$$

determines a canonical section s of K^* . In four dimensions, the vanishing of this section forces the type of a twisted generalized complex structure to jump from 0 to 2. The jump occurs along a 2-torus, which inherits a complex structure.

Definition 5. A point p in the type-changing locus of a twisted generalized complex structure on a 4-manifold is a *nondegenerate point* if it is a nondegenerate zero of $s \in C^\infty(K^*)$.

Regarding the notion of submanifolds in the generalized complex setting, we have the following.

Definition 6. *Branes.* Let (M, H, \mathcal{J}) be a twisted generalized complex manifold, and let $i : \Sigma \hookrightarrow M$ be a submanifold with a 2-form $F \in \Omega^2(\Sigma)$ that satisfies $dF = i^*H$, and such that the sub line bundle $\tau_F \subset (TM \oplus T^*M)|_\Sigma$ defined as

$$\tau_F := \{X + \xi \in T\Sigma \oplus T^*M : i^*\xi = i_X F\},$$

is invariant under \mathcal{J} , i.e., τ_F is a complex sub-bundle. Such a submanifold Σ is called a *brane*.

In the case where M is a complex manifold, the definition of a brane coincides with the notion of a complex submanifold. Analogously, for symplectic manifolds, Lagrangian submanifolds provide examples of branes.

3. TOOLS AND TECHNIQUES OF CONSTRUCTION

The surgical methods used in the production of twisted generalized complex structures on 4-manifolds are introduced in this section.

3.1. Torus surgeries. Let T be a 2-torus of self intersection zero inside a 4-manifold X . There is a diffeomorphism $N_T \rightarrow T^2 \times D^2$ from its tubular neighborhood N_T to a thick 2-torus $T^2 \times D^2$. Let $\{\alpha, \beta\}$ be the generators of $\pi_1(T)$ and consider the meridian μ_T of T inside $X - T$, and the push offs S_α^1, S_β^1 in $\partial N_T \approx T^3$. There is no ambiguity regarding the choice of push offs, since in our constructions the manifold X will be symplectic, the torus T will be Lagrangian, and the push offs are taken with respect to the Lagrangian framing. The loops S_α^1 and S_β^1 are

homologous in N_T to α and β respectively. In particular, the set $\{S_\alpha^1, S_\beta^1, \mu\}$ forms a basis for $H_1(\partial N; \mathbb{Z}) \cong H_1(T^3; \mathbb{Z})$.

The manifold obtained from X by performing a (p, q, r) - torus surgery on T along the curve $\gamma := pS_\alpha^1 + qS_\beta^1$ is defined as

$$X_{T,\gamma}(p, q, r) = (X - N_T) \cup_\phi (T^2 \times D^2),$$

where the diffeomorphism $\phi : T^2 \times \partial D^2 \rightarrow \partial(X - N_T)$ used to glue the pieces together satisfies $\phi_*([\partial D^2]) = p[S_\alpha^1] + q[S_\beta^1] + r[\mu_T]$ in $H_1(\partial(X - N_T); \mathbb{Z})$.

A few words for the reader to get comfortable with our notation are in order. A $(0, 0, 1)$ -torus surgery on T amounts to carving the 2-torus out, and gluing it back in exactly the same way. Therefore, if one performs a $(0, 0, 1)$ - torus surgery along a torus T on X , one has $X_{T,\gamma}(0, 0, 1) = X$. Whenever $p = 0 = q$, the surgery coefficients $(p, q, r) = (0, 0, 1)$ are said to be trivial.

The Euler characteristic and signature of a 4-manifold are invariant under torus surgeries. If the torus T is essential and the surgery coefficients are nontrivial, then

$$b_1(X_{T,\gamma}(p, q, r)) = b_1(X) - 1 \text{ and } b_2(X_{T,\gamma}(p, q, r)) = b_2(X) - 2.$$

Minding the changes on the fundamental group of the manifolds that undergo surgery, we have the following well-known result. We call the push offs $S_\alpha^1 := m$ and $S_\beta^1 := l$.

Lemma 7. *The fundamental group of the manifold obtained by applying a (p, q, r) - torus surgery to X on the torus T along the curve $m + l$ is given by the quotient*

$$\pi_1(X_{T,\gamma}(p, q, r)) \cong \pi_1(X - T) / N(\mu_T^r m_T^p l_T^q),$$

where $N(\mu_T^r m_T^p l_T^q)$ denotes the normal subgroup generated by $\mu_T^r m_T^p l_T^q$.

Proof. cf. [6, Proof Lemma 4]. We argue in terms of the effect that the attachment of n -handles has on the fundamental group. The manifold $T^2 \times D^2$ has a handlebody decomposition consisting of one 0-handle, two 1-handles, and one 2-handle. Using the dual decomposition, in order to glue $T^2 \times D^2$ back in, one attaches one 2-handle, two 3-handles, and one 4-handle. The fundamental group of the resulting manifold changes as in the statement of the lemma by attaching the 2-handle. Attaching 3- and 4-handles has no effect on the fundamental group of the resulting manifold. \square

Provided that certain hypothesis on the manifold that undergoes surgery and on the corresponding torus hold, a geometric structure is readily available for $X_{T,\gamma}(p, q, r)$. If X admits a symplectic form for which the torus T is Lagrangian, then performing a $(p, q, 1)$ -torus surgery on the preferred Lagrangian framing of N_T results in $X_{T,\gamma}(p, q, 1)$ being symplectic [4]; this surgical procedure is known as *Luttinger surgery* [25, 4]. The next section is devoted to describe the existence of twisted generalized complex structures on $X_{T,\gamma}(p, q, 0)$, on its blow ups and blow downs.

Given that the coefficients (p, q, r) already encode the surgery curve γ , from now on it will be dropped from our notation: $X_T(p, q, r) := X_{T,\gamma}(p, q, r)$.

3.2. Surgical procedures to endow manifolds with a generalized complex structure. In [9] and [10], Cavalcanti and Gualtieri have studied and employed classic topological procedures in order to equip a 4-manifold with a twisted generalized complex structure.

Their main results that will be used for our purposes are recalled in the following three theorems.

Theorem 8. [9, Theorem 3.1 and Corollary 1], [10, Theorem 4.1] *(p, q, 0)-torus surgery. Let (M, ω) be a symplectic 4-manifold, which contains a symplectic torus T of self-intersection zero and of area A . Let $\hat{M} := M_T(p, q, 0)$ be the result of performing a $(p, q, 0)$ -torus surgery to M along T . Then \hat{M} admits a twisted generalized complex structure such that*

- *The complex locus is given by an elliptic curve Σ with modular parameter $\tau = i$, and the induced holomorphic differential Ω has periods $< A, iA >$.*
- *Integrability holds with respect to a 3-form H , which is Poincaré dual to A times the homology class of an integral circle of $\text{Re}(\Omega^{-1})$ in Σ .*

The $(p, q, 0)$ -torus surgery can be performed simultaneously on a collection of disjoint symplectic tori in M .

Each time such a surgery is performed, a type change locus is obtained in the twisted generalized complex 4-manifold $M_T(p, q, 0)$. The generalized complex structures produced by Cavalcanti and Gualtieri ([9, 10]) on $m\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ are obtained by applying one $(p, q, 0)$ surgery to an elliptic surface $E(m-1)$ along a torus fiber.

Much like in the symplectic and complex scenarios, the existence of a generalized complex structure on a manifold is preserved under the blow up/blow down operations. The changes on the ambient manifold are exactly the same as in the complex/symplectic case.

Theorem 9. [9, Theorem 3.3] *Blow ups. For any non-degenerate complex point $p \in M$ in a twisted generalized complex 4-manifold M , there exists a twisted generalized complex 4-manifold \tilde{M} and a generalized complex holomorphic map $\pi : \tilde{M} \rightarrow M$ that is an isomorphism $\tilde{M} - \pi^{-1}(\{p\}) \rightarrow M - \{p\}$. The pair (\tilde{M}, π) is called the blow up of M at p , and it is unique up to canonical isomorphism.*

Theorem 10. [9, Theorem 3.4] *Blow downs. A twisted generalized complex 4-manifold \tilde{M} containing a 2-brane $\Sigma = S^2$ intersecting the complex locus in a single non-degenerate point may be blown down to a generalized complex 4-manifold M . That is, there is a generalized holomorphic map $\pi : \tilde{M} \rightarrow M$ to a twisted generalized complex manifold M that is an isomorphism $\tilde{M} - \Sigma \rightarrow M - \{p = \pi(\Sigma)\}$.*

3.3. On the Seiberg-Witten invariants of the manifolds manufactured.

A purpose of this paper is to enlarge the class of twisted generalized complex 4-manifolds that are neither symplectic nor complex. We wish to make sure that the manifolds constructed have trivial Seiberg-Witten invariant [29]. From the work of Taubes [28], such manifolds will not admit a symplectic structure.

The basic result for such a purpose is the adjunction inequality.

Theorem 11. *Adjunction inequality [24, 26]. Let X be a smooth closed oriented 4-manifold with $b_2^+(X) > 1$, and let $\Sigma_g \hookrightarrow X$ be a smoothly embedded genus g surface with non-negative self-intersection. If $g \geq 1$, then every basic class K of X satisfies*

$$2g - 2 \geq | \langle K, [\Sigma_g] \rangle | + [\Sigma_g] \cdot [\Sigma_g].$$

Furthermore, if $g = 0$ and $[\Sigma_g] \in H_2(X; \mathbb{Z})$ is not a torsion class, then the Seiberg-Witten invariant of X vanishes.

4. GENERALIZED COMPLEX STRUCTURES ON SPIN MANIFOLDS

In this section, we occupy ourselves with equipping the almost-complex connected sums of copies of $S^2 \times S^2$ with generalized complex structures. The generalized complex structures built on $S^2 \times S^2$ are new, and we observe some unexpected phenomena concerning the number of type changing loci. Our constructions owe plenty to the efforts of Akhmedov, Baldridge, Fintushel, Kirk, D. Park, and Stern in [7, 12, 3] to construct exotic simply connected 4-manifolds with small Euler characteristics.

4.1. More generalized complex structures on $S^2 \times S^2$. Perturbing the Kähler structure on $\mathbb{CP}^1 \times \mathbb{CP}^1$ yields a generalized complex structure on $S^2 \times S^2$, which has a single type change locus. In this section we to construct different generalized complex structures on $S^2 \times S^2$, in the sense that they have more than one type change loci. We are indebted to Gil Cavalcanti and to Marco Gualtieri for pointing this out.

Consider the product of two copies of a genus 2 surface endowed with the product symplectic structure $\Sigma_2 \times \Sigma_2$. Denote the loops generating the fundamental group $a_i \times \{s_2\}, b_i \times \{s_2\}, \{s_1\} \times c_i, \{s_1\} \times d_i$ by a_1, b_1, a_2, b_2 and c_1, d_1, c_2, d_2 , so that $\pi_1(\Sigma_2) = \langle a_1, b_1, a_2, b_2 | [a_1, b_1][a_2, b_2] = 1 \rangle$ for the first factor, and analogously $\pi_1(\Sigma_2) = \langle c_1, d_1, c_2, d_2 | [c_1, d_1][c_2, d_2] = 1 \rangle$ for the second factor. The topological properties that will be used are summarized in the following proposition.

Proposition 12. (cf. [12, Section 4]). *The first homology group is given by $H_1(\Sigma_2 \times \Sigma_2; \mathbb{Z}) \cong \mathbb{Z}^8$, and it is generated by the loops $a_1, b_1, a_2, b_2, c_1, d_1, c_2$, and d_2 . The second homology group $H_2(\Sigma_2 \times \Sigma_2; \mathbb{Z}) = \mathbb{Z}^{18}$ is generated by the sixteen tori $a_i \times c_j, a_i \times d_j, b_i \times c_j, b_i \times d_j$ ($i = 1, 2, j = 1, 2$), and the surfaces $\Sigma_2 \times \{pt\}, \{pt\} \times \Sigma_2$. The tori $a_i \times c_j, b_i \times c_j$ and the surface $\Sigma_2 \times \{pt\}$ are geometrically dual to the tori $b_i \times d_j, a_i \times d_j$ and the surface $\{pt\} \times \Sigma_2$ respectively, in the sense that $a_i \times c_j$ and $b_i \times d_j$ intersect transversally at one point (and similarly for the other surfaces), and every other intersection is pairwise empty. The intersection form over the integers is even, and it is generated by nine hyperbolic summands.*

The Euler characteristic and signature of the product manifold are given by $e(\Sigma_2 \times \Sigma_2) = 4$, and $\sigma(\Sigma_2 \times \Sigma_2) = 0$.

Notice that the element in the first homology group and the loop generating it are being denoted by the same symbol. More homotopical information on these loops will be needed in our constructions, and we will proceed as follows. Removing a surface from a 4-manifold may change the fundamental group of the ambient

manifold, since non nullhomotopic loops might be introduced by the meridian of the surface, thus adding a generator to the presentation of the group.

According to Fintushel, D. Park and Stern [12, Section 4] inside the complement in $\Sigma_2 \times \Sigma_2$ of eight of the Lagrangian tori described above, one can choose basepaths from a basepoint (x, y) of $\Sigma_2 \times \Sigma_2$ to a basepoint of the boundaries of the tubular neighborhoods of the tori so that the two Lagrangian push offs, and the meridians of the tori are given by

$$\begin{aligned} &\{\tilde{a}_1, \tilde{c}_1; \mu_1 = [\tilde{b}_1^{-1}, \tilde{d}_1^{-1}]\}, \{\tilde{a}_1, \tilde{c}_2; \mu_2 = [\tilde{b}_1^{-1}, \tilde{d}_2^{-1}]\}, \{\tilde{a}_2, \tilde{c}_1; \mu_3 = [\tilde{b}_2^{-1}, \tilde{d}_1^{-1}]\}, \\ &\{\tilde{a}_2, \tilde{c}_2; \mu_4 = [\tilde{b}_2^{-1}, \tilde{d}_2^{-1}]\}, \{\tilde{b}_1, \tilde{d}_1 \tilde{c}_1 \tilde{d}_1^{-1}; \mu_5 = [\tilde{a}_1^{-1}, \tilde{d}_1]\}, \{\tilde{b}_2, \tilde{d}_2 \tilde{c}_2 \tilde{d}_2^{-1}; \mu_6 = [\tilde{a}_2^{-1}, \tilde{d}_2]\}, \\ &\{\tilde{b}_2 \tilde{a}_2 \tilde{b}_2^{-1}, \tilde{d}_1; \mu_7 = [\tilde{b}_2, \tilde{c}_1^{-1}]\}, \{\tilde{b}_1 \tilde{a}_1 \tilde{b}_1^{-1}, \tilde{d}_2; \mu_8 = [\tilde{b}_1, \tilde{c}_2^{-1}]\}. \end{aligned}$$

The elements with tildes denote loops that are homotopic to the corresponding loop that generates an element in homology. In [12, Section 4], homotopies among the loops are found in order to obtain a presentation for the fundamental group of complement of the tori inside $\Sigma_2 \times \Sigma_2$ that is still generated by the same number of generators that $\pi_1(\Sigma_2 \times \Sigma_2)$ has.

Remark 13. *Abuse of notation.* From now on we will make no distinction between a loop, its homotopy class, and the corresponding generator in homology. In particular the decorations above will be abandoned: a loop \tilde{a} will now just be denoted by a . These oversimplifications are justified by following and building upon recent papers [6, 12, 7, 3], where torus surgeries are used to unveil exotic smooth structures on almost-complex 4-manifolds. In particular, we are helped greatly by the analysis done by Baldridge and Kirk in [6, 7].

Proposition 12 says that the torus, its meridian and its Lagrangian pushoffs available for surgery are given by

$$\begin{aligned} T_1 &:= a_1 \times c_1, m_1 = a_1, l_1 = c_1, \mu_1 = [b_1^{-1}, d_1^{-1}], \\ T_2 &:= a_1 \times c_2, m_2 = a_1, l_2 = c_2, \mu_2 = [b_1^{-1}, d_2^{-1}], \\ T_3 &:= a_2 \times c_1, m_3 = a_2, l_3 = c_1, \mu_3 = [b_2^{-1}, d_1^{-1}], \\ T_4 &:= a_2 \times c_2, m_4 = a_2, l_4 = c_2, \mu_4 = [b_2^{-1}, d_2^{-1}], \\ T_5 &:= b_1 \times c_1, m_5 = b_1, l_5 = d_1 c_1 d_1^{-1}, \mu_5 = [a_1^{-1}, d_1], \\ T_6 &:= b_2 \times c_2, m_6 = b_2, l_6 = d_2 c_2 d_2^{-1}, \mu_6 = [a_2^{-1}, d_2], \\ T_7 &:= a_2 \times d_1, m_7 = b_2 a_2 b_2^{-1}, l_7 = d_1, \mu_7 = [b_2, c_1^{-1}], \text{ and} \\ T_8 &:= a_1 \times d_2, m_8 = b_1 a_1 b_1^{-1}, l_8 = d_2, \mu_8 = [b_1, c_2^{-1}]. \end{aligned}$$

The reader is kindly reminded that in our notation, for example, a $(p, 0, r)$ -torus surgery on T_1 stands for a torus surgery on T_1 along the curve $m_1^p = a_1^p$; if $r = 1$, this torus surgery is a Luttinger surgery [4], and the case $r = 0$ correspond to the surgery described in Theorem 8.. We construct generalized complex structures on $S^2 \times S^2$ with a prescribed number of type change loci (up to eight of them) as follows. We exemplify two cases.

Say we would like a generalized complex structure with a number of eight type change loci. Use Gompf's result [14, Lemma 1.6] to perturb the symplectic form on $\Sigma_2 \times \Sigma_2$, so that the eight homologically essential Lagrangian tori $\{T_1, \dots, T_8\}$ become symplectic. Perform simultaneously $(1, 0, 0)$ -surgeries on the tori T_2, T_3, T_5, T_6 , and $(0, 1, 0)$ -surgeries on the tori T_1, T_4, T_7, T_8 . Let $X(8)$ be the

manifold obtained after the surgeries, which is diffeomorphic to $S^2 \times S^2$. By Theorem 8, $X(8)$ admits a generalized complex structure that contains eight type change loci.

Let us now produce a generalized complex structure on $S^2 \times S^2$ that contains four type change loci. Our starting symplectic manifold is again $\Sigma_2 \times \Sigma_2$. Perturb the symplectic form so that the homologically essential Lagrangian tori T_5, T_6, T_7 and T_8 become symplectic tori [14, Lemma 1.6]. Perform four Luttinger surgeries: $(0, 1, +1)$ -surgery on T_1 , $(1, 0, -1)$ -surgery on T_2 , $(1, 0, +1)$ -surgery on T_3 , $(0, 1, -1)$ -surgery on T_4 , and $(1, 0, +1)$ -surgeries on T_5 and on T_6 . Now perform four $(p, q, 0)$ -torus surgeries on the remaining symplectic tori as follows. Simultaneously perform $(1, 0, 0)$ -surgery on T_5 and on T_6 , and $(0, 1, 0)$ -surgery on T_7 and on T_8 . Denote the resulting manifold by $X(4)$.

By Theorem 8, $X(4)$ admits a generalized complex structure. The group $\pi_1(X)$ is generated by the elements $a_1, b_1, a_2, b_2, c_1, d_1, c_2, d_2$, and the following (among others) relations hold has the following presentation:

$$\begin{aligned} c_1 &= [d_1^{-1}, b_1^{-1}], a_1 = [b_1^{-1}, d_2^{-1}], a_2 = [d_1^{-1}, b_2^{-1}], c_2 = [b_2^{-1}, d_2^{-1}], \\ b_1 &= [d_1, a_1^{-1}] = 1, b_2 = [d_2, a_2^{-1}] = 1, d_1 = [c_1^{-1}, b_2] = 1, d_2 = [c_2^{-1}, b_1] = 1. \end{aligned}$$

It is straightforward to see that $\pi_1(X(4)) = \{1\}$. Torus surgeries preserve both the Euler characteristic and the signature. Thus, $e(X(4)) = 4$ and $\sigma(X(4)) = 0$. Moreover, the generalized complex manifold X is spin. Indeed, the homological effect of such a $(p, 0, 0)$ or $(0, q, 0)$ -torus surgery is to kill both the homology class of the torus on which the surgery is performed, and the class of its dual torus. Therefore, the intersection form over the integers changes by a hyperbolic summand, and it remains to be even. Freedman's Theorem [13] implies that the resulting manifold $X(4)$ is homeomorphic to $S^2 \times S^2$. The four type change loci arise as the core tori of the last four torus surgeries.

Inside the symplectic 4-manifold obtained by applying the four Luttinger surgeries are the symplectically imbedded surfaces of genus two $\Sigma_2 \times \{x\}$, and $\{x\} \times \Sigma_2$. The $(p, 0, 0)$ - and $(0, q, 0)$ -torus surgeries reduce the genus of these surfaces, which after the surgeries become imbedded 2-spheres. In particular, $X(4)$ is diffeomorphic to $S^2 \times S^2$.

Variations on the surgical procedure described above yield the following theorem

Theorem 14. *Let $n \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. The 4-manifold $S^2 \times S^2$ admits a generalized complex structure $(X(n), H, \mathcal{J})$, which has n type change loci.*

4.2. The almost-complex connected sums $(2g - 3)(S^2 \times S^2)$, $g \geq 3$. We generalize the procedure described in the previous section with the purpose of endowing the connect sums of $S^2 \times S^2$ that are almost-complex with a generalized complex structure. The first step is to consider the product $\Sigma_2 \times \Sigma_g$ of a genus 2 surface with a genus $g \geq 3$ surface equipped with the product symplectic form. Let a_i, b_i, c_j and d_j ($i = 1, 2, j = 1, \dots, g$) be the standard generators of $\pi_1(\Sigma_2)$ and $\pi_1(\Sigma_g)$ respectively. We encode the topological information we need in the following proposition. In the notation regarding the fundamental group of the manifolds that undergo

surgery, each relation is associated to the torus on which a surgery introduces the aforementioned relation.

Proposition 15. (cf. [3, Section 2]). *The first homology group is given by $H_1(\Sigma_2 \times \Sigma_g; \mathbb{Z}) = \mathbb{Z}^{4+2g}$, and it is generated by the loops $a_1, b_1, a_2, b_2, c_1, d_1, \dots, c_g, d_g$. The second homology group $H_2(\Sigma_2 \times \Sigma_g; \mathbb{Z}) = \mathbb{Z}^{8g+2}$ is generated by $8g$ tori, and the surfaces $[\Sigma_2 \times \{pt\}]$, $[\{pt\} \times \Sigma_g]$: the tori $a_i \times c_j$ and $a_i \times d_j$ are geometrically dual to $b_i \times d_j$ and $b_i \times c_j$ respectively, and the surface $\Sigma_2 \times \{pt\}$ is geometrically dual to $\{pt\} \times \Sigma_g$. The intersection form over the integers is even, and it is given by $4g + 1$ hyperbolic summands; the pairs of tori contribute $4g$ summands, and one summand is contributed by the surfaces $\Sigma_2 \times \{pt\}$ and $\{pt\} \times \Sigma_g$. The characteristic numbers are given by $e(\Sigma_2 \times \Sigma_g) = 4g - 4$, and $\sigma(\Sigma_2 \times \Sigma_g) = 0$.*

The manifold X_g with $b_1(X_g) = 0$ obtained from applying $4 + 2g$ (p, q, r) -torus surgeries to $\Sigma_2 \times \Sigma_g$ ($p, q \in \{0, 1\}, p \neq q$) has fundamental group generated by the elements $a_1, b_1, a_2, b_2, c_1, d_1, \dots, c_g, d_g$, and the following relations hold.

$$\begin{aligned} T_1 : a_1 &= [b_1^{-1}, d_1^{-1}]^r, T_2 : b_1 = [a_1^{-1}, d_1]^r, T_3 : a_2 = [b_2^{-1}, d_2^{-1}]^r, T_4 : b_2 = [a_2^{-1}, d_2]^r \\ T_5 : c_1 &= [d_1^{-1}, b_2^{-1}]^r, T_6 : d_1 = [c_1^{-1}, b_2]^r, T_7 : c_2 = [d_2^{-1}, b_1^{-1}]^r, T_8 : d_2 = [c_2^{-1}, b_1]^r \\ T_9 : c_3 &= [a_1^{-1}, d_3^{-1}]^r, T_{10} : d_3 = [a_2^{-1}, c_3^{-1}]^r, \dots, T_{3+2g} : c_g = [a_1^{-1}, d_g^{-1}]^r, \\ T_{4+2g} : d_g &= [a_2^{-1}, c_g^{-1}]^r. \end{aligned}$$

The proof is a generalization of [12, Section 4]. The claims concerning the submanifolds, and their homological properties are left to the reader. A proof for the nontrivial claim regarding the fundamental group calculations can be found in [21]. Notice that in the notation of Proposition 15, the relation introduced to the fundamental group by the corresponding (p, q, r) -surgery appears associated to its corresponding homologically essential torus. For example, $T_1 : a_1 = [b_1^{-1}, d_1^{-1}]^1$ stands for the relation coming from a $(1, 0, 1)$ -surgery. The $(1, 0, 0)$ -surgery of Theorem 8 applied to T_1 is then $T_1 : a_1 = [b_1^{-1}, d_1^{-1}]^0 = 1$.

We proceed to construct a generalized complex structure on the connected sums $(2g - 3)(S^2 \times S^2)$ for $g \geq 3$. Using [14, Lemma 1.6], we can perturb the symplectic form on $\Sigma_2 \times \Sigma_g$ so that the tori $\{T_1, T_2, T_3, \dots, T_{4+2g}\}$ become symplectic [14, Lemma 1.6]. Let X_g be the manifold of Proposition 15 obtained by applying torus surgeries with $r = 0$. Theorem 8 implies that X_g admits a generalized complex structure.

By the presentation of the group $\pi_1(X_g)$ given in Proposition 15, this implies that the manifold X_g is simply connected. A direct computation of the characteristic numbers yields $e(X_g) = e(\Sigma_2 \times \Sigma_g) = -2 \cdot (2 - 2g) = 4g - 4$, and $\sigma(X_g) = \sigma(\Sigma_2 \times \Sigma_g) = 0$. By Freedman's Theorem [13], X_g has the homeomorphism type of $(2g - 3)(S^2 \times S^2)$.

We are ready to prove the following result.

Theorem 16. *Let $m \geq 1$. The connected sum*

$$m(S^2 \times S^2)$$

admits a generalized complex structure if and only if it admits an almost-complex structure.

In particular, $m = (2g - 3)$ is an odd number. The necessity of the existence of an almost-complex structure is clear [16]. Concerning the sufficiency of the condition, it remains to be demonstrated that the manifold X_g constructed above is $(2g - 3)(S^2 \times S^2)$.

Proof. Theorem 8 implies X_g admits a generalized complex structure. We claim that X_g is diffeomorphic to $(2g - 3)(S^2 \times S^2)$. For the sake of clarity, we work out the case $g = 3$ by looking at the effect the surgeries have on the submanifolds of $\Sigma_2 \times \Sigma_3$; the proofs for the cases $g > 3$ follow a natural generalization. The torus surgeries that were performed on $\Sigma_2 \times \Sigma_3$ transform the imbedded surfaces $\Sigma_2 \times \{x\}$ and $\{x\} \times \Sigma_3$ into spheres, by reducing their genus. Moreover, according to Proposition 15, there are four remaining homologically essential submanifolds that are geometrically dual. Inside $\Sigma_2 \times \Sigma_3$, these are tori, which after the surgeries become geometrically dual 2-spheres. For example, the $(1, 0, 0)$ -surgery on T_1 reduces the genus of the tori $a_1 \times c_2, a_1 \times c_3$, and the genus of their dual tori as well. Thus, becoming imbedded 2-spheres. Each of these pairs span an $S^2 \times S^2$ summand in X_g . Therefore, X_3 is diffeomorphic to $3(S^2 \times S^2)$. The Seiberg-Witten invariant of X_3 is trivial, and it is not symplectic. It is not a complex manifold by Kodaira's classification of complex surfaces [22, 8]. \square

Remark 17. *Number of loci.* Let $g \geq 3$. The generalized complex structures built in Theorem 16 have a number of $4 + 2g$ type change loci. Following the method described in Section 4.1, generalized complex structures with a different number of loci can be produced.

4.3. Preview of other fundamental groups. Constructions of non-simply connected 4-manifolds are given in Section 6. However, At this stage we would like to point out that a variation on the coefficients of the torus surgeries in the previous constructions promptly yields twisted generalized complex 4-manifolds with several other fundamental groups. A sample for abelian groups is the following.

Proposition 18. *Let $g \geq 2$. The homeomorphism types corresponding to*

- $\pi_1 = \mathbb{Z}/p\mathbb{Z} : (2g - 3)(S^2 \times S^2) \# \widetilde{L(p, 1)} \times S^1$
- $\pi_1 = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z} : (2g - 1)(S^2 \times S^2) \# \widetilde{L(p, 1)} \times S^1$
- $\pi_1 = \mathbb{Z} : (2g)(S^2 \times S^2) \# S^3 \times S^1$

admit a twisted generalized complex structure, which does not come from a symplectic nor a complex structure.

The pieces $\widetilde{L(p, 1)} \times S^1$ and $\widetilde{L(p, 1)} \times S^1$ stand for the manifolds obtained by modifying the product $L(p, 1) \times S^1$ of a Lens space with a circle as follows. Perform a surgery on $L(p, 1) \times S^1$ along $\{x\} \times \alpha$ ($x \in L(p, 1)$) to kill the loop corresponding to the generator of the infinite cyclic group factor so that $\pi_1 = \mathbb{Z}/p\mathbb{Z}$ of the resulting manifold comes from the fundamental group of the Lens space. This amounts to removing a neighborhood of the loop $S^1 \times D^3$ and glueing in a $S^2 \times D^2$. Denote such a manifold by $\widetilde{L(p, 1)} \times S^1$. Applying the same procedure to the loop $\{x\} \times \alpha^q$ results in a 4-manifold with $\pi_1 = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}$. Such manifold is denoted by $\widetilde{L(p, 1)} \times S^1$.

Proof. These manifolds are constructed by changing the surgery coefficients on one or two of the torus surgeries used in the proof of Theorem 16 following Lemma 7. Generalized complex manifolds with infinite cyclic fundamental group are built by *not* performing one of the torus surgeries. Twisted generalized complex manifolds with finite cyclic fundamental group are built by changing one of the surgery coefficients $(1, 0, 0)$ for a $(p, 0, 0)$ with $p \neq 0$. The homeomorphism criteria in the infinite cyclic fundamental group case is due to Hambleton and Teichner [19, Corollary 3]. If $p > 1$, then one obtains a twisted generalized complex manifold with finite cyclic fundamental groups, and its universal cover is nonspin. The 3-form H is given by the generator of H^3 . The corresponding homeomorphism criteria is due to Hambleton and Kreck, and it is given in [18, Theorem C]. The case $\pi_1 = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}$ is left as an exercise; the homeomorphism criteria, also due to Hambleton and Kreck, is given in [18, Theorem B].

The argument minding the lack of a symplectic or a complex structure is verbatim to the one given for Theorem 16. \square

5. CONSTRUCTIONS OF NON-SYMPLECTIC AND NON-COMPLEX GENERALIZED COMPLEX MANIFOLDS THROUGH THE ASSEMBLAGE OF SYMPLECTIC ONES

As an immediate corollary to the main result of the previous section, we construct new generalized complex structures on non-spin simply connected almost-complex 4-manifolds. We also recover an existence result of Cavalcanti and Gualtieri.

Proposition 19. (cf. [10, Section 5]). *Let $n \geq m$. The manifolds*

$$m\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$$

admit a generalized complex structure if and only if they admit an almost-complex one. Moreover, there exist generalized complex structures on $m\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ with more than one type change loci.

In particular, m is an odd number. We set $m = 2g - 3$. Blowing up (Theorem 9) and/or blowing down (Theorem 10) $(2g - 3)(S^2 \times S^2)$ produces generalized complex structures on $(2g - 3)\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ that can be chosen to have as many as $4 + 2g$ type change loci. As it was mentioned in Remark 17, the construction can be modified to obtain a different number of loci (cf. Section 4.1). For the sake of simplicity, we will give the argument for the case $g = 3$; the other cases follow verbatim. The symbol $X = Y$ indicates a diffeomorphism between manifolds X and Y .

Proof. The necessity of the existence of an almost complex structure is clear [16]. Let X_3 be the blow up of the generalized complex manifold $3(S^2 \times S^2)$. Theorem 9 implies X_3 is generalized complex. We have

$$\begin{aligned} X_3 &= 3(S^2 \times S^2) \# \overline{\mathbb{CP}^2} = 2(S^2 \times S^2) \# (S^2 \times S^2 \# \overline{\mathbb{CP}^2}) = \\ &= 2(S^2 \times S^2) \# \mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2} = (S^2 \times S^2 \# \overline{\mathbb{CP}^2}) \# (S^2 \times S^2 \# \overline{\mathbb{CP}^2}) \# \mathbb{CP}^2 = \\ &= (\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}) \# (\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}) \# \mathbb{CP}^2 = 3\mathbb{CP}^2 \# 4\overline{\mathbb{CP}^2}. \end{aligned}$$

An iteration of the usage of blow ups and blow downs (Theorems 9 and 10) allows us to conclude that the manifold $3\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ admits a generalized complex structure. \square

For the remaining part of the section we proceed to consider more general production schemes of generalized complex structures on manifolds that are neither symplectic nor complex, by taking a symplectic manifold as a starting point. The following constructions are motivated by [27].

Proposition 20. *Assume $h \geq 1$. Let X be a simply connected symplectic 4-manifold that contains a symplectic surface of genus two $\Sigma \subset X$ that satisfies $[\Sigma]^2 = 0$, and $\pi_1(X - \Sigma) = 1$. The manifolds*

$$Z_{X,h} := X \# 2h(S^2 \times S^2) \text{ and} \\ Z'_{X,h} := X \# 2h(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2})$$

admit a generalized complex structure.

Proof. We work out the case $h = 1$; the other cases follow verbatim from the argument. Consider the manifold $\Sigma_2 \times T^2$ endowed with the product symplectic form. Let a_1, b_1, a_2, b_2 be the generators of $\pi_1(\Sigma_2)$, and x, y the generators of $\pi_1(T^2)$. It is easy to see that this manifold contains four pairs of homologically essential Lagrangian tori, and a symplectic surface of genus two. The tori are displayed below; the genus two surface is a parallel copy of the surface $\Sigma_2 \times \{pt\}$, and we will continue to call it Σ_2 during the proof. According to Baldrige and Kirk [7, Proposition 7], the fundamental group

$$\pi_1(\Sigma_2 \times T^2 - (\Sigma_2 \cup T_1 \cup \dots \cup T_4))$$

is generated by the loops x, y, a_1, b_1, a_2, b_2 . Moreover, with respect to certain paths to the boundary of the tubular neighborhoods of the T_i and Σ_2 , the meridians and two Lagrangian push offs are given by

- $T_1 : m_1 = x, l_1 = a_1, \mu_1 = [b^{-1}, y^{-1}]$,
- $T_2 : m_2 = y, l_2 = b_1 a_1 b^{-1}, \mu_2 = [x^{-1}, b_1]$,
- $T_3 : m_3 = x, l_3 = a_2, \mu_3 = [b_2^{-1}, y^{-1}]$,
- $T_4 : m_4 = y, l_4 = b_2 a_2 b_2^{-1}, \mu_4 = [x^{-1}, b_2]$,
- $\mu_{\Sigma_2} = [x, y]$.

The loops a_1, b_1, a_2, b_2 lie on the genus 2 surface and form a standard set of generators; $[a_1, b_1][a_2, b_2] = 1$ holds.

Build the symplectic sum [14, Theorem 1.3]

$$Z := X \#_{\Sigma=\Sigma_2} \Sigma_2 \times T^2.$$

Given that the loops a_1, b_1, a_2, b_2 lie on $\Sigma_2 \times \{x\} \subset \Sigma_2 \times T^2$ for $x \in T^2$, and the meridian is given by $\mu_{\Sigma_2} = [x, y]$, using van-Kampen's Theorem we see that our hypothesis $\pi_1(X - \Sigma) = 1$ implies $\pi_1(Z) = \mathbb{Z}x \oplus \mathbb{Z}y$. Perturb the symplectic form so that the tori $\{T_1, T_2, T_3, T_4\}$ become symplectic [14, Lemma 1.6]. Perform simultaneously a $(1, 0, 0)$ -surgery on each of the tori to obtain a manifold $Z_{X,2}$. By Theorem 8, the manifold $Z_{X,2}$ admits a generalized complex structure, which has four type change loci. The surgeries set $x = 1 = y$. Thus, $\pi_1(Z_{X,2}) = 1$. To conclude on the diffeomorphism type of $Z_{X,2}$, we observe the effect that the torus surgeries has on the embedded submanifolds. The tori $y \times b_2$ and $y \times a_2$ become embedded 2-spheres, each of which is a factor of an $S^2 \times S^2$ summands. Thus, $Z_{X,2}$ is diffeomorphic to $X \# 2(S^2 \times S^2)$. In order to construct $Z_{X,h}$ for $h \geq 2$, one builds the symplectic sum

$$Z := X \#_{\Sigma=\Sigma_2} \Sigma_2 \times \Sigma_h$$

and submits it to surgical procedure similar to the one described above, making use of Proposition 15. The generalized complex manifold $Z'_{X,h}$ is obtained by blowing up at a nondegenerate point on $Z_{X,h}$ (Theorem 9), and then blowing it down (Theorem 10). \square

To finalize this section, we produce generalized complex structures on more general connected sums.

Proposition 21. *Suppose $h \geq 2$. Let X and Y be symplectic 4-manifolds that contain symplectic tori $T_X \subset X, T_Y \subset Y$ such that $[T_X]^2 = 0 = [T_Y]^2$, and $\pi_1(X) = \pi_1(X - T_X) = \pi_1(Y) = \pi_1(Y - T_Y) = 1$. The manifolds*

$$Z_{X,Y,2h-1} := X \# (2h-1)(S^2 \times S^2) \# Y \text{ and}$$

$$Z'_{X,Y,2h-1} := X \# (2h-1)(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}) \# Y$$

admit a generalized complex structure.

The following proof was inspired by an argument due to Z. Szabó given in [27].

Proof. Following Proposition 12, perturb the symplectic sum [14, Lemma 1.6] so that the tori $\{T_1, T_4, T_5, T_6, T_7, T_8\}$ become symplectic. Build the symplectic sum [14, Theorem 1.3]

$$Z := X \#_{T_X=T_1} \Sigma_2 \times \Sigma_2 \#_{T_4=T_Y} Y.$$

Notice that in $\pi_1(Z)$, we have $a_1 = c_1 = a_2 = c_2 = 1$. Simultaneously perform $(0, 1, 0)$ -surgeries on the tori T_5, T_6, T_7 , and T_8 . Denote the resulting generalized complex 4-manifold by $Z_{X,Y,3}$ (Theorem 8). This manifold contains four type change loci.

Let us take a look at the surviving submanifolds of $Z_{X,Y,3}$. The torus surgeries performed along b_1 and b_2 turn the tori $b_1 \times d_1, b_2 \times d_1$, and the genus 2-surface $\{x\} \times \Sigma_2$ into imbedded 2-spheres of self-intersection zero. Each of these spheres is a factor of an $S^2 \times S^2$ -summand. Therefore,

$$Z_{X,Y,3} = X \# 3(S^2 \times S^2) \# Y.$$

The generalized complex manifold constructed is neither symplectic [28] nor complex [22, 8]. The generalized complex manifold $Z'_{X,Y,3} = X \# 3(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}) \# Y$ is obtained by blowing up a nondegenerate point on $Z_{X,h}$ (Theorem 9), and then blowing it down (Theorem 10). \square

Remark 22. *Submanifolds of higher genus.* The reader will notice that the symplectic sums involved in the previous arguments can be modified in order to yield similar results in the case the starting manifold X contains symplectic submanifolds of higher genus.

6. NON-SIMPLY CONNECTED TWISTED GENERALIZED COMPLEX 4-MANIFOLDS

Given that our principal mechanism of construction is to apply torus surgeries to non-simply connected building blocks, an organic next step is the study of existence of twisted generalized complex structures in the non-simply connected realm.

6.1. Finitely presented fundamental groups. In his lovely paper [14], Gompf proved that every finitely presented group is the fundamental group of a symplectic 4-manifold. Given that the existence of a symplectic structure naturally induces the existence of a generalized complex structure (Example 4), *a posteriori* the fundamental group imposes no restriction on the existence of such a structure.

Moreover, Kotschick [23] proved that any finitely presented group occurs as the fundamental group of an almost-complex 4-manifold. It is an interesting question, how much the size of a generalized complex 4-manifold depends on its fundamental group. In this direction we obtain the following result.

Theorem 23. *Let G be a finitely presented group with a presentation consisting of g generators x_1, \dots, x_g , and r relations w_1, \dots, w_r , and let k be a nonnegative integer. There exists a non-symplectic twisted generalized complex 4-manifold $X(G, k)$ with*

$$\pi_1(X(G, k)) \cong G, \quad e(X(G, k)) = 4(g + r + 2) + k, \quad \text{and} \quad \sigma(X(G, k)) = -k.$$

We point out that the manifolds of Theorem 23 do not share the homotopy type of a complex surface [8]. The remaining part of the section is technical; see also Remark 24.

In order to prove the theorem, we follow an argument due to Baldridge and Kirk. In [5], these authors build a symplectic manifold N , which serves as a fundamental building block for these constructions since it allows the manipulation on the number of generators and relations on fundamental groups. We proceed to describe it.

Begin with a 3-manifold Y that fibers over the circle, and build the 4-manifold

$$N := Y \times S^1.$$

Its Euler characteristic and its signature are both zero, and it admits a symplectic structure [5, p. 856].

The fundamental group of N has the following presentation.

$$\pi_1(N) = \langle H, t \mid R_*^g(x) = txt^{-1}, x \in H \rangle \times \langle s \rangle,$$

where the group H has a presentation given by

$$H = \langle x_{1,1}, y_{1,1}, \dots, x_{g,1}, y_{g,1}, x_{1,2}, y_{1,2}, \dots, x_{g,2}, y_{g,2}, \dots, x_{1,n}, y_{1,n}, \dots, x_{g,n}, y_{g,n} \mid \prod_{i=1}^n \prod_{j=1}^g [x_{j,i}, y_{j,i}] \rangle.$$

Let G be a finitely presented group with g generators and r relations. The fundamental group $\pi_1(Y \times S^1)$ has classes $s, t, \gamma_1, \dots, \gamma_{g+r}$ so that

$$G \cong \pi_1(Y \times S^1) / N(s, t, \gamma_1, \dots, \gamma_{g+r}),$$

where $N(s, t, \gamma_1, \dots, \gamma_{g+r})$ is the normal subgroup generated by the aforementioned classes.

The symplectic manifold N contains $g + r + 1$ symplectic tori

$$T_0^Y, T_1^Y, \dots, T_{g+r}^Y \subset N$$

of self-intersection zero that have two practical features regarding our fundamental group computations.

- The generators of $\pi_1(T_0^Y)$ represent s and t , and
- the generators of $\pi_1(T_i^Y)$ represent s and γ_i .

The curve s has the form $\{y\} \times S^1 \subset Y \times S^1$, with $y \in Y$, and γ_i has the form $\gamma_i \times \{x\} \subset Y \times \{x\}$, $x \in S^1$. The role of the curves γ_i is to provide the r relations in a presentation of G [5, Section 4]; we point out that the number of curves γ_i is greater than r .

Serious technical issues arise during cut-and-paste constructions in the computation of fundamental groups, the choice of basepoint being the culprit. The argument used here to prove Theorem 23 builds upon the careful analysis done by J. Yazinski in [30, Section 4].

Let R be the symplectic manifold obtained from $\Sigma_2 \times \Sigma_2$ by applying the following five Luttinger surgeries. Perform two $(1, 0, -1)$ -surgeries on the tori T_5, T_6 , and three $(0, 1, -1)$ -surgeries on the tori T_3, T_7, T_8 . The manifold R contains the homologically essential Lagrangian tori $T_1, T_2, T_4 \subset R$. Perturb its symplectic form so that all these tori become symplectic [14, Lemma 1.6].

Let us pin down a basepoint for the fundamental group computations that are to come. Set the basepoint to be $v \in \partial(\nu(T_1))$, where $\nu(T_1)$ is a tubular neighborhood of T_1 . The fundamental group $\pi_1(R - T_1)$ is generated by the elements $\{a_1, b_1, \dots, c_2, d_2, \epsilon_1, \epsilon_2, \dots, \epsilon_m\}$, where the elements $\{\epsilon_j\}$ are conjugates to the based meridian $\mu_1 = [b_1^{-1}, d_1^{-1}]$. Let β be a path inside $R - T_1$ from the basepoint (x, y) that was chosen to calculate π_1 of the complement of the tori inside $\Sigma_2 \times \Sigma_2$ (cf. [12] and Proposition 12) to v , which is contained in the torus $b_1 \times d_1$ and such that it intersects T_1 transversally at one point. The paths representing generators of $\pi_1(\Sigma_2 \times \Sigma_2, (x, y))$ can be conjugated by β in order to obtain paths that are based at v . The relations stated in Proposition 12 continue to hold in $\pi_1(R - T_1, v)$ under the new choice of basepoint.

Proof. The task at hand is to construct a twisted generalized complex manifold $X(G)$ that is not symplectic, such that $\pi_1(X(G)) = G$, $e(X(G)) = 4(g + r + 2)$, and $\sigma(X(G)) = 0$. This is done by applying torus surgeries to a symplectic sum along tori composed of Y , a symplectic manifold built from $\Sigma_2 \times \Sigma_3$, and $g + r + 1$ copies of R , which we will denote by R_i , $i = 1, \dots, g + r$. The symplectic tori inside each copy are denoted by $T_{1,i}, T_{2,i}, T_{4,i} \subset R_i$. Once $X(G)$ is obtained, applying blow ups to it results in the manifold $X(G, k)$. The details are as follows.

Consider the manifold $\Sigma_2 \times \Sigma_3$ equipped with the product symplectic form. A small modification on the fundamental group computations in [21] and Proposition 15 yields that the Lagrangian tori inside $\Sigma_2 \times \Sigma_3$, the meridians and the Lagrangian pushoffs of twelve tori are given by

$$\begin{aligned} T_1 &:= a_1 \times c_1, m_1 = a_1, l_1 = c_1, \mu_1 = [b_1^{-1}, d_1^{-1}], \\ T_2 &:= a_1 \times c_2, m_2 = a_1, l_2 = c_2, \mu_2 = [b_1^{-1}, d_2^{-1}], \\ T_3 &:= a_1 \times c_3, m_3 = a_1, l_3 = c_3, \mu_3 = [b_1^{-1}, d_3^{-1}], \\ T_4 &:= a_2 \times c_1, m_4 = a_2, l_4 = c_1, \mu_4 = [b_2^{-1}, d_1^{-1}], \\ T_5 &:= a_2 \times c_2, m_5 = a_2, l_5 = c_2, \mu_5 = [b_2^{-1}, d_2^{-1}], \\ T_6 &:= a_2 \times c_3, m_6 = a_2, l_6 = c_3, \mu_6 = [b_2^{-1}, d_3^{-1}], \\ T_7 &:= b_1 \times c_1, m_7 = b_1, l_7 = d_1 c_1 d_1^{-1}, \mu_7 = [a_1^{-1}, d_1], \\ T_8 &:= b_2 \times c_2, m_8 = b_2, l_8 = d_2 c_2 d_2^{-1}, \mu_8 = [a_2^{-1}, d_2], \end{aligned}$$

$$\begin{aligned}
T_9 &:= a_2 \times d_1, m_9 = b_2 a_2 b_2^{-1}, l_9 = d_1, \mu_9 = [b_2, c_1^{-1}], \\
T_{10} &:= a_1 \times d_2, m_{10} = b_1 a_1 b_1^{-1}, l_{10} = d_2, \mu_{10} = [b_1, c_2^{-1}], \\
T_{11} &:= a_1 \times d_3, m_{11} = b_1 a_1 b_1^{-1}, l_{11} = d_3, \mu_{11} = [b_1, c_3^{-1}], \text{ and} \\
T_{12} &:= a_2 \times d_3, m_{12} = b_2 a_2 b_2^{-1}, l_{12} = d_3, \mu_{12} = [b_2, c_3^{-1}].
\end{aligned}$$

Construct a symplectic manifold S by applying Luttinger surgeries and perturbing the symplectic form as follows. Perform $(1, 0, -1)$ -surgery on the tori $T_1, T_2, T_4, T_5, T_7, T_8$, and on the tori T_9, T_{10}, T_{11} perform $(0, 1, -1)$ -surgeries. By [4], the resulting manifold is symplectic. Now, perturb the symplectic form so that the tori $T_3, T_{12} \subset S$ become symplectic [14, Lemma 1.6]. Notice that $H_1(S; \mathbb{Z}) \cong \mathbb{Z}$ is generated by c_3 .

Regarding $\pi_1(S - T_{12})$, set the basepoint to be $v_{12} \in \partial(\nu(T_{12}))$. Let β_{12} be a path in $S - \nu(T_{12})$ from the chosen basepoint (x_2, y_3) of $\pi_1(\Sigma_2 \times \Sigma_3, (x_2, y_3))$ used to compute the fundamental group of the complement of the tori to v_{12} , which is contained in the torus $b_2 \times c_3$ and intersects T_{12} transversally in one point. Conjugating the paths that represent generators of $\pi_1(\Sigma_2 \times \Sigma_3, (x_2, y_3))$ by β_{12} yields paths based at v_{12} . The fundamental group $\pi_1(S - T_{12})$ is generated by the elements $\{a_1, b_1, \dots, c_3, d_3, \epsilon_1^S, \dots, \epsilon_m^S\}$, where the elements $\{\epsilon_j^S\}$ are conjugates to the based meridian μ_{12} .

Following [14] and [5], we start by building a symplectic manifold that provides us with the generators of G (plus some generators that we will be getting rid of at a later stage with the usage of torus surgeries), and that contains enough submanifolds to introduce the relations of G in a later step. For these purposes, build the symplectic sum [14, Theorem 1.3]

$$Z := S \#_{T_{12}=T_0^Y} Y$$

using the diffeomorphism $\phi : T_{12} \rightarrow T_0^Y$ that induces the identifications $a_2 \mapsto s$, $d_3 \mapsto t$ on fundamental groups. The meridian μ_0^Y is identified with the based meridian $\mu_{12} = [b_2, c_3^{-1}]$. Let z be the basepoint for the block Z , and let η be a path that takes z to $\phi(v_{12})$ with $v_{12} \in \partial(\nu(T_{12}))$. Conjugation by η provides us with paths that represent elements in π_1 . We abuse notation, and we keep calling the conjugated elements with the same symbols.

Using Seifert-van Kampen's Theorem, the fundamental group is given by

$$\pi_1(Z) = \frac{\pi_1(Y - \nu(T_0^Y)) * \pi_1(S - \nu(T_{12}))}{\langle a_2 s^{-1}, d_3 t^{-1}, \mu_0^Y (\mu_{12})^{-1} \rangle}.$$

Here, $\nu(T_0^Y), \nu(T_{12})$ denote tubular neighborhoods of the corresponding tori; in particular, these manifolds intersect in an open neighborhood of T^3 .

Moreover, the tori T_1^Y, \dots, T_{r+g}^Y are contained in Z . The next step is to use these tori to perform $r + g$ symplectic sums with the purpose of introducing the r relations in the presentation of G . In the process, more relations are introduced into the group presentation. We will get rid of any extra relation by using torus surgeries at a latter step. Take $i = 1, \dots, r + g$ copies of R , and call each copy R_i . We start by choosing based curves in R_i that are homotopic to the generators $a_{1,i}$, and $c_{1,i}$ for all i . By Section 4.1, the based meridian $\mu_{1,i}$ is taken to be the commutator $[b_{1,i}^{-1}, d_{1,i}^{-1}]$. These push offs are contained in the boundary of the tubular neighborhood of the respective torus inside R_i .

Minding the choice of base point for the application of Seifert-van Kampen's Theorem, we will follow the discussion that precedes the proof for each copy of R . So, we have basepoints $v_i \in \partial(\nu(T_{1,i}))$ and conjugating paths β_i .

Let $Z' := Z - (\nu(T_1^Y) \cup \dots \cup \nu(T_{r+g}^Y))$, where $\nu(T_i^Y)$ denotes a tubular neighborhood of the symplectic torus $T_i^Y \subset Z$. Build the symplectic sums of Z and R_i along $T_{1,i}$ using the diffeomorphism $\phi_i : T_{1,i} \rightarrow T_i^Y$ that identify the generators

$$\begin{aligned} a_{1,i} &\mapsto s_i \\ c_{1,i} &\mapsto \gamma_i \end{aligned}$$

to build a symplectic manifold \hat{X} . The tori $T_{2,i}, T_{4,i}, T_3'$ are contained in \hat{X} . Simultaneously perform $(0, 1, 0)$ -, $(1, 0, 0)$ -, and $(0, 1, 0)$ -surgery respectively on these tori to set $c_{2,i} = a_{2,i} = c_3 = 1$. Denote the resulting twisted generalized complex manifold by $X(G)$. We claim that the fundamental group of $X(G)$ is given by the finitely presented group G of our hypothesis.

We need to make sure that the usage of Seifert-van Kampen's Theorem is done carefully. There must be a common basepoint that is being shared by the open neighborhoods of the pieces being glued together. Set z_0 to be the basepoint in Z' . Regarding the basepoints $v_i \in \partial(\nu(T_{1,i}))$ for the pieces $R_i - T_{1,i}$, fix paths η_i that take z_0 to $\phi(v_{12})$. Take $\overline{R_i - \nu(T_{1,i})}$ to be the union of $R_i - \nu(T_{1,i})$ and a neighborhood of the path η_i . We continue to abuse notation, and denote an element of $\pi_1(\overline{R_i - \nu(T_{1,i})})$ with the same symbol for the related element in $\pi_1(R_i - \nu(T_{1,i}))$, since conjugation with the path η_i correlates the elements and the relations continue to hold.

By Seifert-van Kampen's Theorem we have

$$\pi_1(X(G)) \cong \frac{\pi_1(Z') * \pi_1(\overline{R_1 - \nu(T_{1,1})}) * \dots * \pi_1(\overline{R_{r+g} - \nu(T_{1,r+g})})}{N},$$

where N is the normal subgroup of $\pi_1(Y) * \pi_1(S) * \pi_1(R_1 - T_{1,1}) * \dots * \pi_1(R_{r+g} - T_{1,r+g})$ generated by $a_{1,i} \bar{s}_i^{-1}, c_{1,i} \bar{\gamma}_i^{-1}$, the meridians $\mu'_{12} \mu_0^Y, \mu_i^Y \mu_{1,i}^{-1}$ and their conjugates, and $c_{2,i} = a_{2,i} = c_3 = 1$. The element $\bar{\gamma}_i$ is a based loop in Z' that is freely homotopic to a push off of the loop $\gamma_i \times \{x\} \subset \gamma_i \times S^1$ to the boundary of a tubular neighborhood of the torus $T_i = \gamma_i \times S^1$. The element μ_i^Y is a based meridian corresponding to the torus $T_i \subset Z'$. Given that the curves \bar{s}_i and s are of the form $\{x\} \times S^1$, the elements \bar{s}_i are based curves in Z' that are homotopic to s on the boundary of a tubular neighborhood of $T_i = \gamma_i \times S^1$ (cf. [30, Proof of Lemma 4.3]).

We need to see that the elements $a_1, b_1, a_2, b_2, c_1, d_1, c_2, d_2, d_3^{-1}t, b_{1,i}, b_{2,i}, d_{1,i}, d_{2,i}$ (and the conjugates of the meridians $\epsilon_{1,i}, \dots, \epsilon_{n,i}, \epsilon_1^S, \dots, \epsilon_m^S$) for $i = 1, \dots, g+r$ are trivial in $\pi_1(X(G))$. The $(1, 0, 0)$ - and $(0, 1, 0)$ -torus surgeries kill the generators $b_{1,i}, b_{2,i}, d_{1,i}, d_{2,i}$. This can be seen by plugging in any of the identities $c_{2,i} = 1 = a_{2,i}$ in the presentation of $\pi_1(R_i)$ (see Proposition 12). Moreover, the triviality of the three generators $c_{2,i} = a_{2,i} = c_3 = 1$ implies that the meridians $\mu_{12}, \mu_0^Y, \mu_i^Y, \mu_{1,i}$ and their conjugates are trivial in $\pi_1(X(G))$. Using the identity $c_3 = 1$ in the relations in $\pi_1(S)$, we see that $a_1 = b_1 = a_2 = b_2 = c_1 = d_1 = c_2 = d_2 = d_3 = 1$, and $t = 1$ due to the identifications of the generators of the tori during the construction of the symplectic sum Z . Moreover, since $\mu_i^Y = 1$ for $i = 1, \dots, r+g$, their conjugates are trivial. This implies that $\bar{\gamma}_i$ is conjugate to γ_i in $\pi_1(X(G))$. By the choice of

γ_i [5], we have $\pi_1(X(G)) \cong G$.

To argue that the manifold $X(G)$ does not admit a symplectic structure, we proceed as follows. The torus T'_6 is contained in $X(G)$, since it was disjoint from all the tori involved in the surgeries. The $(0, 1, 0)$ -surgery that was applied on T'_3 turns T'_6 into an embedded 2-sphere of self-intersection zero. By the Adjunction inequality (Theorem 11), the manifold $X(G)$ has trivial Seiberg-Witten invariants. Taubes' work [28] implies that $X(G)$ does not admit a symplectic structure, as claimed.

The computations of the characteristic numbers of $X(G)$ are straight-forward. \square

Remark 24. *Another construction of a manifold $X(G)$.* A twisted generalized complex 4-manifold that is neither symplectic, nor complex with arbitrary finitely presented fundamental group, Euler characteristic $10 + 4(g + r + 1)$ and signature -1 can be produced as follows. The symplectic manifold in the main result of [30] contains a symplectic 2-torus whose meridian in the complement is trivial. Use this symplectic 2-torus to form a symplectic sum [14] of the manifold X in [30, Theorem 1.1] with $\Sigma_2 \times \Sigma_2$, and then apply accordingly six surgeries of Theorem 8 cf. Section 4.1. Such a twisted generalized complex structure contains six type change locus cf. Remark 17..

6.2. Abelian groups. It is of our interest to provide explicit constructions, which we hope will be useful in improving our understanding of twisted generalized complex structures. We begin this section with two examples that expose different twisted generalized complex structures on the same 4-manifold.

Example 25. *Twisted generalized complex structure on $S^3 \times S^1$ with a single type change locus* (cf. [17, Example 6.38], [10, Example 4.1]). Consider $T^2 \times S^2$ endowed with the product symplectic form. A $(1, 0, 0)$ -torus surgery on the symplectic submanifold $T^2 \times \{s\}$ for $s \in S^2$ along one of the loops carrying a generator of $\pi_1(T^2 \times S^2)$ results in $S^3 \times S^1$. The diffeomorphism is established as follows. The handlebody of $T^2 \times S^2$ is given at the top of Figure 1. We are using the 1-handle notation introduced by Akbulut in [1]. The effect that a $(1, 0, 0)$ -torus surgery has on the handlebody decomposition of $T^2 \times D^2$ is to interchange a 1-handle (dotted circle) and a 0-framed 2-handle [15, Figure 8.25, p.316], [2, Figure 17]. Thus, performing a $(0, 0, 1)$ -torus surgery on $T^2 \times \{pt\}$ results in the second diagram of Figure 1. The second and third rows show the isotopies and handle cancellations that establish a diffeomorphism with $S^3 \times S^1$. By Theorem 8, this primary Hopf surface has a twisted generalized complex structure with one type change locus.

Example 26. *Twisted generalized complex structure on blow-ups of $L(p, 1) \times S^1$ with one, two or three type change loci.* To build a twisted generalized complex structure on this secondary Hopf surface, start with the 4-torus $T^4 = T^2 \times T^2$ endowed with the product symplectic form. Denote by x, y, a, b both the generators of $\pi_1(T^2 \times T^2) = \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}a \oplus \mathbb{Z}b$ and the corresponding loops as well. Under this notation, the tori

$$T_1 := x \times a, T_2 := y \times a$$

and their respective dual tori $y \times b, x \times b$ are Lagrangian, and the torus $T_3 := a \times b$ and its dual $x \times y$ are symplectic.

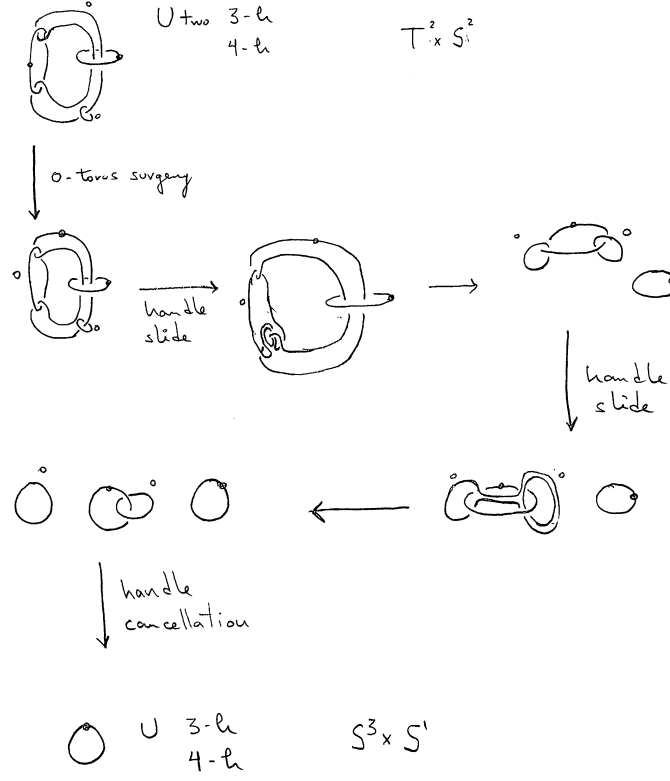


FIGURE 1. The result of performing a $(1,0,0)$ -surgery to $T^2 \times S^2$ on the 2-torus $T^2 \times \{pt\}$ is $S^3 \times S^1$

According to Baldridge and Kirk [6, Section 2] and [7, Theorem 1], the fundamental group of $T^4 - (T_1 \cup T_2)$ is generated by the loops x, y, a, b and the relations $[x, a] = [y, a] = 1$ hold. The meridians of the tori and the two Lagrangian push offs of their generators are given by the following formulae:

$$m_1 = x, l_1 = a, \mu_1 = [b^{-1}, y^{-1}] \text{ and}$$

$$m_2 = y, l_2 = bab^{-1}, \mu_2 = [x^{-1}, b].$$

Apply a $(p, 0, -1)$ -torus surgery on T_1 . This Luttinger surgery introduces the relation $x^p = [b^{-1}, y^{-1}]$. Now, do a $(1, 0, 1)$ -surgery on T_2 to introduce the relation $y = [x^{-1}, b]$. Last, perform a $(0, 1, 0)$ -torus surgery on T_3 along b , and call the resulting manifold $X_p(1)$. The last surgery sets $b = 1$, which implies $y = [x^{-1}, 1] = 1$, and $x^p = 1$. Since the elements x and a commute, we have

$$\pi_1(X_p(1)) = \langle x, a : x^p = 1, [x, a] = 1 \rangle \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}.$$

Theorem 8 implies that $X_p(1)$ admits a twisted generalized complex structure with one single type change locus. If we simultaneously perform $(1, 0, 0)$ -surgery on a symplectic T_2 , and $(0, 1, 0)$ -torus surgery on T_3 , a twisted generalized complex structure $(X_p(2), H, \mathcal{J})$ with two type change loci. Performing simultaneously the surgery of Theorem 8 on three symplectic T_1, T_2, T_3 yields a third twisted generalized complex structure $(X_p(3), H, \mathcal{J})$ with three type change loci.

Set $X_p := X_p(i)$ for $i \in \{1, 2, 3\}$. Let us check that there exists a diffeomorphism $X_p \rightarrow L(p, 1) \times S^1$, by using the fact that a diffeomorphism between two 3-manifolds extends to a diffeomorphism between the 4-manifolds obtained by taking the product of S^1 with the diffeomorphic 3-manifolds. We are grateful to Ronald J. Stern, having learned the following argument from him.

Think of the 4-torus as $T^4 = T^3 \times S^1 = (x \times y \times b) \times a$. The torus surgeries performed to obtain X_p can be seen as Dehn surgeries on $D^2 \times T^2 = (D^2 \times S^1) \times S^1$ along the curves x, y, b in T^3 (see Section 3.1). In particular, we have that the twisted generalized complex manifold $X_p = \widetilde{T^3} \times S^1$, where $\widetilde{T^3}$ is the 3-manifold obtained from applying the three Dehn surgeries on the 3-torus.

The diffeomorphism between $\widetilde{T^3}$ and $L(p, 1)$ can be seen through the analysis of the effect that the Dehn surgeries have on the handlebody of T^3 . The 3-torus is obtained by 0-surgery on the Borromean rings [15, Figure 5.25, p. 159], and the Lens spaces are obtained by performing a $-p$ -surgery on the unknot [15, Figure 5.24, p.158]. Thus, X_p is diffeomorphic to $L(p, 1) \times S^1$.

By blowing up k non-degenerate points and iterating the usage of Theorem 9, one concludes that

$$L(p, 1) \times S^1 \# k \overline{\mathbb{CP}^2}$$

admits three twisted generalized complex structures with one, two, and three type change loci.

Remark 27. *New twisted generalized complex structures on blow-ups of $S^3 \times S^1$ and $T^2 \times S^2$.* Since there are diffeomorphisms $L(1, 1) = S^3$ and $L(0, 1) = S^1 \times S^2$, Example 26 covers the existence of twisted generalized complex structures on blow-ups of $S^3 \times S^1$ and on blow ups of $T^2 \times S^2$ as well. In particular, the twisted generalized complex structure on $S^3 \times S^1$ has two type change loci. The twisted generalized complex structure on the Hopf surface displayed in Example 25 has a single type change locus. Furthermore, it was proven above that there are twisted generalized complex structures on $T^2 \times S^2$ with one or two type change loci.

The main result of this section is the following theorem.

Theorem 28. *The manifolds*

- $\hat{m}(S^2 \times S^2) \# S^3 \times S^1$, and
- $\hat{m}\mathbb{CP}^2 \# \hat{n}\mathbb{CP}^2 \# S^3 \times S^1$

admit a twisted generalized complex structure if and only if they admit an almost complex structure.

The production of these manifolds involve torus surgeries on the product $\Sigma_2 \times \Sigma_g$ of a genus two surface and a genus g surface, as it was done for Theorem 16 in Section 4.2. In particular $\hat{m} = m + 1 = 2g - 2$. For the number of type change loci see Remark 17 and Section 4.1.

Proof. The necessity of the existence of an almost-complex structure is clear [16]. The construction of the manifolds claimed by the first item in the statement of the theorem was carried out in Proposition 18. To see that these manifolds are reducible, one proceeds as in the proof of Theorem 16. The existence of the manifolds of the second claim in the statement follows by applying blow ups and blow downs (Theorem 9 and Theorem 10) to the manifolds of the first item (compare with the proof of Proposition 19). \square

The symmetric product of a genus g surface $Sym^2(\Sigma_g)$ is a surface of general type obtained by taking the quotient of the product of two surfaces of genus g $\Sigma_g \times \Sigma_g$ by the action of the involution $\Sigma_g \times \Sigma_g \rightarrow \Sigma_g \times \Sigma_g$ defined by $(x, y) \mapsto (y, x)$. By applying the procedure described in the previous sections to this Kähler manifold one obtains the following proposition.

Proposition 29. *Let k_1, k_2, k_3, k_4 be a nonnegative integers. There are twisted generalized complex non-symplectic, non-complex 4-manifolds with for the following homeomorphism classes*

- $\pi_1 = 1 : (g^2 - 3g + 1)\mathbb{CP}^2 \# k_1 \overline{\mathbb{CP}^2}$,
- $\pi_1 = \mathbb{Z}/p\mathbb{Z} : (g^2 - 3g + 1)\mathbb{CP}^2 \# k_2 \overline{\mathbb{CP}^2} \# \widetilde{L(p, 1)} \times S^1$
- $\pi_1 = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z} : (g^2 - 3g + 1)\mathbb{CP}^2 \# k_3 \overline{\mathbb{CP}^2} \# \widetilde{L(p, 1)} \times S^1$
- $\pi_1 = \mathbb{Z} : (g^2 - 3g + 2)\mathbb{CP}^2 \# k_4 \overline{\mathbb{CP}^2} \# S^3 \times S^1$

We finish the section by mentioning that it is straight-forward to obtain similar existence results of twisted generalized complex manifolds that do not admit a symplectic nor a complex structure, and whose fundamental group is among the following choices

$$\pi_1 = \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_{2g-1}\mathbb{Z}.$$

6.3. Free and surface groups. For what follows we start with the symplectic manifold $T^2 \times \Sigma_g$ endowed with the product form.

Theorem 30. *Let $g \geq 2$, and assume k to be a nonnegative integer. There exist twisted generalized complex non-symplectic 4-manifolds $X_{F,g,k}(i)$ and $X_{S,g,k}(j)$, which have $e(X_{F,g,k}) = e(X_{S,g,k}) = k$ and $\sigma(X_{F,g,k}) = \sigma(X_{S,g,k}) = -k$ such that*

- $\pi_1(X_{F,g,k}(i)) = \overbrace{\mathbb{Z} * \cdots * \mathbb{Z}}^g$
- $\pi_1(X_{S,g,k}(j)) = \pi_1(\Sigma_g)$.

The twisted generalized complex structures on these manifolds can be chosen to have a number of $i \in \{1, \dots, 2 + g\}$ and $j \in \{1, 2\}$ type change loci respectively.

The starting point of the construction is the manifold $T^2 \times \Sigma_g$, the product of a 2-torus and a surface of genus g , equipped with the product symplectic form. We gather the topological properties of this manifold that will be used in the following result.

Lemma 31. *The first homology group $H_1(T^2 \times \Sigma_g; \mathbb{Z}) \cong \mathbb{Z}^{2g+2}$ is generated by $x, y, a_1, b_1, \dots, a_g, b_g$. The second homology $H_2(T^2 \times \Sigma_g; \mathbb{Z}) \cong \mathbb{Z}^{4g+2}$ is generated by $T^2 \times \{s_g\}$ ($\{s_g\} \in \Sigma_g$), $\{t\} \times \Sigma_g$ ($\{t\} \in T^2$), and the tori $x \times a_i, y \times a_i, x \times b_i, y \times b_i$ where $i = 1, 2, \dots, g$. The fundamental group $\pi_1(T^2 \times \Sigma_g - \bigcup_{i=1}^g T_i)$ is generated by the elements $x, y, a_1, b_1, \dots, a_g, b_g$, and the relations*

$$\begin{aligned} [x, a_i] &= [y, a_i] = [y, b_i a_i b_i^{-1}] = 1 \text{ for } i = 1, \dots, g, \text{ and} \\ [x, y] &= 1 = [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] \end{aligned}$$

hold in this group. The tori with the corresponding meridian, and Lagrangian push offs are given by

- $T_1 : m_1 = x, l_1 = a_1, \mu_1 = [b_1^{-1}, y^{-1}],$
- $T_2 : m_2 = y, l_2 = b_1 a_1 b_1^{-1}, \mu_2 = [x^{-1}, b_1],$
- $T_3 : m_3 = x, l_3 = a_2, \mu_3 = [b_2^{-1}, y^{-1}],$
- $T_4 : m_4 = y, l_4 = b_2 a_2 b_2^{-1}, \mu_4 = [x^{-1}, b_2],$
- $T_5 : m_5 = x, l_5 = a_3, \mu_5 = [b_3^{-1}, y^{-1}],$
- \vdots
- $T_{2g-1} : m_{2g-1} = x, l_{2g-1} = a_{g-1}, \mu_{2g-1} = [b_{g-1}^{-1}, y^{-1}], \text{ and}$
- $T_{2g} : m_g = y, l_g = b_g a_g b_g^{-1}, \mu_{2g} = [x^{-1}, b_g].$

The proof of this lemma is omitted. It is an straight-forward generalization of [7, Proposition 7]; compare with Proposition 15. The proof for both instances of the theorem consists of building a generalized complex 4-manifold with trivial Euler characteristic, trivial signature, and with the desired fundamental group. Then one concludes by blowing up k points on such a manifold (Theorem 9). Let us start with the case of surface groups.

Proof. (Theorem 22). Perturb the product symplectic form of $T^2 \times \Sigma_g$ [14, Lemma 1.6] so that T_1 and T_2 become symplectic. Denote by $X_{S,g}(2)$ the twisted generalized complex 4-manifold obtained from $T^2 \times \Sigma_g$ by applying $(1, 0, 0)$ -surgeries on T_1 and on T_2 . Theorem 8 implies that $X_{S,g}(2)$ admits a twisted generalized complex structure with two type change loci. We have $e(X_{S,g}(2)) = 0 = \sigma(X_{S,g}(2))$. Indeed, the Euler characteristic and the signature remain invariant under torus surgeries, so $e(X_{S,g}(2)) = e(T^2 \times \Sigma_g) = 0$ and $\sigma(X_{S,g}(2)) = \sigma(T^2 \times \Sigma_g) = 0$. Moreover, the surgeries set $x = 1 = y$. Thus,

$$\pi_1(X_{S,g}(2)) = \pi_1(\Sigma_g).$$

The manifold $X_{S,g,k}(2)$ is a twisted generalized complex 4-manifold obtained by blowing up $X_{S,g}(2)$ at k non-degenerate points (Theorems 8 and 9). A straight-forward computation yields $e(X_{S,g,k}(2)) = k, \sigma(X_{S,g,k}(2)) = -k$. Since blow ups do not alter the fundamental group, we have $\pi_1(X_{S,g,k}(2)) = \pi_1(\Sigma_g)$. The twisted generalized complex structure with one type change locus is built by doing $(1, 0, 1)$ -surgery on the Lagrangian T_1 , and a $(1, 0, 0)$ -surgery on the symplectic T_2 .

Let us consider the case of free fundamental groups of rank g . We claim that there exists a twisted generalized complex 4-manifold $X_{F,g}(i)$ with i type change

loci, with trivial Euler characteristic, trivial signature, and whose fundamental group is generated by b_1, b_2, \dots, b_g so that

$$\pi_1(X_{F,g}(i)) = \mathbb{Z}b_1 * \dots * \mathbb{Z}b_g.$$

The manifold $X_{F,g,k}(i)$ is obtained by blowing up $X_{F,g}(i)$ at k non-degenerate points (Theorem 9). As in the previous case, by a straight-forward computation, one checks $e(X_{F,g,k}(i)) = k$, $\sigma(X_{F,g,k}(i)) = -k$, and

$$\pi_1(X_{F,g,k}) = \overbrace{\mathbb{Z} * \dots * \mathbb{Z}}^g.$$

The twisted generalized complex manifold $X_{F,g}(i)$ is constructed as follows. We work out the case of one single type change locus ($i = 1$); the other cases follow from a small variation of the argument as in the surface groups case that was discussed before. Perturb the symplectic form on $T^2 \times \Sigma_g$ so that the torus T_2 becomes symplectic. Perform $(0, 1, -1)$ -surgeries on T_1 , $(1, 0, -1)$ -surgery on T_3 , and $(0, 1, -1)$ -surgeries on T_j for $4 \leq j \leq g + 2$. Perform $(1, 0, 0)$ -surgery on T_2 . The relation induced by the last surgery is $y = 1$; by using this on the relations introduced by the Luttinger surgeries, one sees $x = a_1 = a_2 = \dots = a_g = 1$. This implies

$$\pi_1(X_{F,g}(1)) = \mathbb{Z}b_1 * \dots * \mathbb{Z}b_g.$$

By Theorem 8, $X_{F,g}(1)$ is a twisted generalized complex manifold, and it has trivial Euler characteristic and signature zero.

The triviality of the Seiberg-Witten invariants of any manifold constructed above follows from Theorem 11, by keeping track of the submanifolds on Lemma 31 and observing the existence of an imbedded 2-sphere of self-intersection zero. \square

7. ON FURTHER RESEARCH

We finish the paper with two interesting questions.

Question 32. *What are the sufficient conditions for a manifold to admit a generalized complex structure in dimension four?*

Question 33. *What is the relation between an almost-complex structure and a generalized complex structure on a given smooth 4-manifold?*

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REFERENCES

- [1] S. Akbulut, *On 2-dimensional homology classes of 4-manifolds*, Math. Proc. Camb. Phil. Soc. 82 (1977), 99-106.
- [2] S. Akbulut and K. Yasui, *Corks, Plugs and exotic structures*, J. Gökova Geom. Topol. Vol. 2 (2008), 40-82.
- [3] A. Akhmedov and B. D. Park, *Exotic smooth structures on small 4-manifolds with odd signatures*, Invent. Math. 181 (2010), no. 3, 577-603.
- [4] D. Auroux, S. K. Donaldson and L. Katzarkov, *Luttinger surgery along Lagrangian tori and non-isotopy for singular symplectic plane curves*, Math. Ann. 326 (2003) 185 - 203 MR1981618.
- [5] S. Baldrige and P. Kirk, *On symplectic 4-manifolds with prescribed fundamental group* Comment. Math. Helv. 82 (2007), no. 4, 845-875.
- [6] S. Baldrige and P. Kirk, *A symplectic manifold homeomorphic but not diffeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$* , Geometry & Topology 12 (2008), 919-940, MR 2403801.
- [7] S. Baldrige and P. Kirk, *Constructions of Small Symplectic 4-Manifolds using Luttinger Surgery*, Journal of Differential Geometry, 82, No. 2 (2009), 317-362.
- [8] W.P. Barth, K. Hulek, C.A.M. Peters and A. van de Ven, *Compact Complex Surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 4, Springer, Berlin, Second Enlarged Edition, 2004.
- [9] G.R. Cavalcanti and M. Gualtieri, *A surgery for generalized complex structures on 4-manifolds*, J. Differential Geom. 76 (2007) no. 1, 35-43.
- [10] G.R. Cavalcanti and M. Gualtieri, *Blow-up of generalized complex 4-manifolds* J. Topology 2 (2009) 840 - 864.
- [11] S. Donaldson, *Seiberg-Witten equations and 4-manifold topology*, Bull. Amer. Math. Soc. 33 (1996), 45-70.
- [12] R. Fintushel, B.D. Park and R. Stern, *Reverse Engineering small 4-manifolds*, Algebraic and Geometric Topology 7 (2007) 2103-2116.
- [13] M. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. 17 (1982), no. 3, 357-453.
- [14] R. E. Gompf, *A new construction of symplectic manifolds*, Ann. of Math. (2) 142 (1995), no. 3, 527-595.
- [15] R. E. Gompf and A. I. Stipsicz, *4-Manifolds and Kirby Calculus*. Graduate Studies in Mathematics, 20. AMS, Providence, RI, 1999.
- [16] M. Gualtieri, *Generalized complex geometry*, Ann. of Math. to appear.
- [17] M. Gualtieri, *Generalized complex geometry*, PhD thesis, Oxford University (2004) arXiv:0401221.
- [18] I. Hambleton and M. Kreck, *Cancellation, Elliptic Surfaces and the Topology of Certain Four-Manifolds*, J. Reine Angew. Math., 444 (1993), 79-100.
- [19] I. Hambleton and P. Teichner, *A Non-Extended Hermitian form over $\mathbb{Z}[\mathbb{Z}]$* , Manuscripta Mathematica 94 (1997), 435 - 442.
- [20] N. Hitchin, *Generalized Calabi-Yau manifolds*, Quart. J. Math. Oxford 54 (2003) 281-308, MR2113140.
- [21] M. C. Hughes, *Branched covering constructions and the symplectic geography problem*, PhD Thesis, University of Waterloo, 2008.
- [22] K. Kodaira, *On the structure of compact complex analytic surfaces*, I, Amer. J. Math. 86 (1964) 751-798. MR0187255.
- [23] D. Kotschick, *All fundamental groups are almost-complex*, Proc. Amer. Math. Soc. 134 (2006), 3081-3083.
- [24] P.B. Kronheimer and T.S. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Res. Lett. 1 (1994), 797 - 808.
- [25] K. M. Luttinger, *Lagrangian Tori in \mathbb{R}^4* , J. Diff. Geom. 42 (1995) 220-228.
- [26] J. W. Morgan, T. S. Mrowka, Z. Szabó, *Product formulas along T^3 for Seiberg-Witten invariants*, Math. Res. Lett. 4 (1997) 915-929 MR1492130
- [27] Z. Szabó, *Simply-connected irreducible 4-manifolds with no symplectic structures*, Invent. Math. 132, (1998), 457-466.
- [28] C. H. Taubes, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Lett. 1 (1994), no.6, 809 - 822.

- [29] E. Witten, *Monopoles and four-manifolds*, Math. Res. Lett. 1 (6) (1994), 769 - 796, MR1306021.
- [30] J. Yazinski, *A new bound on the size of symplectic 4-manifolds with prescribed fundamental group* to appear, J. of Symplectic Geom (2010).

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